COMPACT ALMOST RICCI SOLITONS WITH CONSTANT SCALAR CURVATURE ARE GRADIENT

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Abstract. The aim of this note is to prove that any compact almost Ricci solitons $\big(M^n,\, g,\, X,\, \lambda\big)$ with constant scalar curvature is isometric to a Euclidean sphere $\mathbb{S}^n$. As a consequence we obtain that every compact almost Ricci soliton with constant scalar curvature is gradient. Moreover, the vector field X decomposes as the sum of a Killing vector field Y and the gradient of a suitable function.

1. Introduction and Statement of the results

The study of an almost Ricci soliton was introduced in a recent paper due to Pigola et al. [8], where essentially they modified the definition of Ricci solitons by adding the condition on the parameter λ to be a variable function, more precisely, we say that a Riemannian manifold $(M^n,\, g)$ is an almost Ricci soliton, if there exist a complete vector field X and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying

$$R_{ij} + \frac{1}{2}(X_{ij} + X_{ji}) = \lambda g_{ij}, \tag{1.1}$$

where R_{ij} and $X_{ij} + X_{ji}$ stand, respectively, for the Ricci tensor and the Lie derivative of X in coordinates. We shall refer to this equation as the fundamental equation of an almost Ricci soliton $\big(M^n,\, g,\, X,\, \lambda\big)$. It will be called *expanding*, *steady* or *shrinking*, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Otherwise, it will be called *indefinite*. When the vector field X is a gradient of a smooth function $f : M^n \to \mathbb{R}$ the manifold will be called a gradient almost Ricci soliton. In this case the preceding equation turns out

$$R_{ij} + \nabla^2_{ij} f = \lambda g_{ij}, \tag{1.2}$$

where $\nabla^2_{ij} f$ stands for the Hessian of f.

Moreover, when either the vector field X is trivial, or the potential f is constant, the almost Ricci soliton will be called *trivial*, otherwise it will be a *nontrivial* almost Ricci soliton. We notice that when $n \geq 3$ and X is a Killing vector field an almost Ricci soliton will be a Ricci soliton, since in this case we have an Einstein manifold, from which we can apply Schur's lemma to deduce that λ is constant. Taking into account that the soliton function λ is not necessarily constant, certainly comparison with soliton theory will be modified. In particular, the rigidity result contained in Theorem 1.3 of [8] indicates that almost Ricci solitons should reveal a reasonably broad generalization of the fruitful concept of classical soliton. In fact, we refer the reader to [8] to see some of these changes.

In the direction to understand the geometry of almost Ricci soliton, Barros and Ribeiro Jr. proved in [2] that a compact gradient almost Ricci soliton with nontrivial conformal vector field is isometric to a Euclidean sphere. In the same paper they proved an integral

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formula for compact case, which was used to prove several rigidity results, for more details see [2]. In [5], Catino proved that a locally conformally flat gradient almost Ricci soliton, around any regular point of f, is locally a warped product with $(n-1)$-dimensional fibers of constant sectional curvature.

Example 1. *In the compact case a simple example appeared in* [2]*. It was built over the standard sphere* $\big(\mathbb{S}^n, g_0\big)$ *endowed with the conformal vector field* $X = a^\top$*, where a is a fixed vector in* $\mathbb{R}^{n+1}$ *and* $a^\top$ *stands for its orthogonal projection over* $T\mathbb{S}^n$*. We notice that* $a^\top$ *is the gradient of the right function* h_a*; for more details see the quoted paper.*

It is well-known that all compact 2-dimensional Ricci solitons are trivial, see [7]. However, the previous example gives that there exists a nontrivial compact 2-dimensional almost Ricci soliton. On the other hand, given a vector field X on a compact oriented Riemannian manifold M^n the Hodge-de Rham decomposition theorem, see e.g. [11], gives that we may decompose X as a sum of a gradient of a function h and a free divergence vector field Y, i.e.

$$X = \nabla h + Y, \tag{1.3}$$

where $divY = 0$, see [2]. For simplicity let us call h the Hodge-de Rham potential.

Now we present a generalization to integral formulae obtained in Theorem 4 of [2] for the gradient case.

Theorem 1. *Let* $\big(M^n, g, X, \lambda\big)$ *be a compact almost Ricci soliton. If S and dV_g stand for the scalar curvature and the Riemannian volume form of M^n, respectively, then we have:*

1. $\int_M |Ric - \frac{S}{n}g|^2 dV_g = \frac{n-2}{2n}\int_M \langle\nabla S, X\rangle dV_g = -\frac{n-2}{2n}\int_M S div X dV_g$.
2. $\int_M |Ric - \frac{S}{n}g|^2 dV_g = \frac{n-2}{2n}\int_M \langle\nabla S, \nabla h\rangle dV_g$.

As a consequence of the previous theorem we deduce a strong characterization to any compact almost Ricci soliton with constant scalar curvature, where X is not necessarily the gradient of a potential function f. More precisely, we have the following corollary.

Corollary 1. *Let* $\big(M^n, g, X, \lambda\big)$, $n \geq 3$, *be a non trivial compact almost Ricci soliton. Then, M^n is isometric to a Euclidean sphere $\mathbb{S}^n$ provided:*

1. $\int_M \langle\nabla S, X\rangle dV_g = 0$.
2. $\mathcal{L}_X S = 0$, *where $\mathcal{L}$ denotes Lie derivative.*
3. *S is constant.*
4. *M^n is homogeneous.*

We notice that all above conditions are equivalents. This result try to answers the following problem proposed in [8]:

Problem 1. *Under which conditions a compact almost Ricci soliton is necessarily gradient?*

As a consequence of Corollary 1 we have the following result which answers the previous problem for dimension bigger than two.

Corollary 2. *Every compact almost Ricci soliton with constant scalar curvature is gradient.*

Now we invoke Hodge-de Rham decomposition (1.3) to write

$$\frac{1}{2}\mathcal{L}_X g = \nabla^2 h + \frac{1}{2}\mathcal{L}_Y g. \tag{1.4}$$

In order to answer positively to Problem 1 we may decide whether Y is a Killing vector field. An affirmative answer to this question is given below.

Corollary 3. *Let* $\big(M^n, g, X, \lambda\big), n \geq 3$, *be a non trivial compact almost Ricci soliton. Then on the Hodge-de Rham decomposition Y is a Killing vector field on M^n provided:*

1. *X is a conformal vector field.*
2. *M^n has constant scalar curvature.*

2. Preliminaries

In this section we shall present some preliminaries which will be used during the paper. First we remember that for a Riemannian manifold (M^n, g) it is well-known the following lemma, see [6].

Lemma 1. *Let (M^n, g) be a Riemannian manifold and X a vector field in M. Then*

$$X_{ijk} - X_{ikj} = X_t R_{tijk} \tag{2.1}$$

$$X_{ijkl} - X_{ikjl} = R_{tijk} X_{tl} + R_{tijk,l} X_t \tag{2.2}$$

$$X_{ijkl} - X_{ijlk} = R_{tikl} X_{tj} + R_{tjkl} X_{it} \tag{2.3}$$

Another important proprieties concern to the Ricci tensor and are given below

$$R_{ij,k} = R_{ji,k} \tag{2.4}$$

$$R_{ij,k} - R_{ik,j} = -R_{tijk,t} \tag{2.5}$$

$$R_{ij,kl} - R_{ij,lk} = R_{tikl} R_{tj} + R_{tjkl} R_{it} \tag{2.6}$$

$$\frac{1}{2} S_k = R_{ki,i} = R_{ik,i} \tag{2.7}$$

Using the previous lemma we obtain the following result.

Lemma 2. *Let (M^n, g, X, λ) be an almost Ricci soliton. Then, the following formulae hold:*

$$S + X_{ii} = n\lambda \tag{2.8}$$

$$R_{lj} X_l = -X_{jii} - (n-2)\lambda_j \tag{2.9}$$

$$R_{ij,k} - R_{ik,j} = -\frac{1}{2} R_{lijk} X_l + \frac{1}{2}(X_{kij} - X_{jik}) + \lambda_k g_{ij} - \lambda_j g_{ik} \tag{2.10}$$

Proof. In order to obtain (2.8) it is enough to contract equation (1.1). For equation (2.9) computing the trace of equation (2.1) in i and k we obtain

$$X_{iji} - X_{iij} = R_{liji} X_l = R_{lj} X_l.$$

Next, using fundamental equation (1.1) we have

$$\begin{aligned} R_{ij,i} &= -\frac{1}{2}(X_{iji} + X_{jii}) + \lambda_i g_{ij} \\ &= -\frac{1}{2}(X_{iji} - X_{iij} + X_{iij} + X_{jii}) + \lambda_i g_{ij} \\ &= -\frac{1}{2} R_{lj} X_l - \frac{1}{2}(X_{iij} + X_{jii}) + \lambda_i g_{ij}. \end{aligned}$$

Hence, using the twice contracted second Bianchi identity (2.7) we deduce

$$\frac{1}{2} S_j = -\frac{1}{2} R_{lj} X_l - \frac{1}{2}(X_{iij} + X_{jii}) + \lambda_i g_{ij},$$

which enables us obtain equation (2.9) after comparing the previous expression with covariant derivative of (2.8).

Proceeding, we shall derive equation (2.10). Indeed, taking covariant derivative of (1.1) we deduce

$$R_{ij,k} + \frac{1}{2}(X_{ijk} + X_{jik}) = \lambda_k g_{ij}$$

and

$$R_{ik,j}+\frac{1}{2}(X_{ikj}+X_{kij})=\lambda_j g_{ik}.$$

Now we compare these previous expressions and we use equation (2.1) to obtain

$$\begin{aligned}R_{ij,k}-R_{ik,j} &= -\frac{1}{2}(X_{ijk}+X_{jik}-X_{ikj}-X_{kij})+\lambda_k g_{ij}-\lambda_j g_{ik}\\ &= -\frac{1}{2}R_{lijk}X_l+\frac{1}{2}(X_{kij}-X_{jik})+\lambda_k g_{ij}-\lambda_j g_{ik},\end{aligned}$$

which concludes the proof of the lemma. □

The next lemma is the main result of this section, it will be used to prove Theorem 1.

Lemma 3. *For an almost Ricci soliton* (M^n,g,X,λ) *we have*

$$\begin{aligned}\Delta_X R_{ik} &= 2\lambda R_{ik}-2R_{ijks}R_{js}+\frac{1}{2}R_{is}(X_{sk}-X_{ks})+\frac{1}{2}R_{sk}(X_{si}-X_{is})\\ &+ (n-1)\lambda_{ik}+\lambda_{jj}g_{ki}-\lambda_{ij}g_{kj}.\end{aligned}\tag{2.11}$$

Proof. Using equation (2.10) we obtain

$$R_{ki,j}-R_{kj,i}=\frac{1}{2}R_{lkji}X_l+\frac{1}{2}(X_{jki}-X_{ikj})+\lambda_j g_{ki}-\lambda_i g_{kj},$$

taking the covariant derivative of the previous identity we have

$$R_{ki,jt}-R_{kj,it}=\frac{1}{2}(R_{ijkl,t}X_l+R_{ijkl}X_{lt})+\frac{1}{2}(X_{jkit}-X_{ikjt})+\lambda_{jt}g_{ki}-\lambda_{it}g_{kj}.\tag{2.12}$$

On the other hand, from (2.6) we deduce

$$\begin{aligned}R_{jk,ij} &= R_{jk,ji}+R_{sjij}R_{sk}+R_{skij}R_{js}\\ &= R_{jk,ji}+R_{si}R_{sk}+R_{skij}R_{js}.\end{aligned}\tag{2.13}$$

Next, since $\Delta R_{ik}=R_{ik,jj}$, comparing the previous expression with (2.12) we obtain

$$\Delta R_{ik}=R_{jk,ij}+\frac{1}{2}(R_{ijkl,j}X_l+R_{ijkl}X_{lj})+\frac{1}{2}(X_{jkij}-X_{ikjj})+\lambda_{jj}g_{ki}-\lambda_{ij}g_{kj}.\tag{2.14}$$

Moreover, by second Bianchi identity we have

$$\begin{aligned}R_{ijkl,j}X_l &= -R_{ijlj,k}X_l-R_{ijjk,l}X_l\\ &= -R_{il,k}X_l+R_{ik,l}X_l.\end{aligned}$$

Comparing with (2.14) and using equation (2.13) we have

$$\begin{aligned}\Delta R_{ik} &= R_{jk,ij}+\frac{1}{2}(R_{ik,l}-R_{il,k})X_l+\frac{1}{2}R_{ijkl}X_{lj}\\ &+ \frac{1}{2}(X_{jkij}-X_{ikjj})+\lambda_{jj}g_{ki}-\lambda_{ij}g_{kj}\\ &= R_{jk,ji}+R_{si}R_{sk}+R_{skij}R_{js}+\frac{1}{2}(R_{ik,l}-R_{il,k})X_l\\ &+ \frac{1}{2}R_{ijkl}X_{lj}+\frac{1}{2}(X_{jkij}-X_{ikjj})+\lambda_{jj}g_{ki}-\lambda_{ij}g_{kj},\end{aligned}$$

thus, using the twice contracted Bianchi identity given by (2.7) and (1.1) we obtain

$$\begin{aligned}
\Delta R_{ik} &= \frac{1}{2}S_{ki} + R_{si}R_{sk} + R_{skij}R_{js} - \frac{1}{2}R_{skij}X_{sj} \\
&+ \frac{1}{2}(R_{ik,l} - R_{il,k})X_l + \frac{1}{2}(X_{jkij} - X_{ikjj}) + \lambda_{jj}g_{ki} - \lambda_{ij}g_{kj} \\
&= \frac{1}{2}S_{ki} + R_{si}R_{sk} + R_{skij}R_{js} - R_{skij}(-R_{sj} + \lambda g_{sj} - \frac{1}{2}X_{js}) \\
&+ \frac{1}{2}(R_{ik,l} - R_{il,k})X_l + \frac{1}{2}(X_{jkij} - X_{ikjj}) + \lambda_{jj}g_{ki} - \lambda_{ij}g_{kj} \\
&= \frac{1}{2}S_{ki} + R_{si}R_{sk} + 2R_{skij}R_{js} + \lambda R_{ik} + \frac{1}{2}R_{skij}X_{js} \\
&+ \frac{1}{2}(R_{ik,s} - R_{is,k})X_s + \frac{1}{2}(X_{jkij} - X_{ikjj}) + \lambda_{jj}g_{ki} - \lambda_{ij}g_{kj}.
\end{aligned} \tag{2.15}$$

Next, we computate the following sum

$$Y = \frac{1}{2}S_{ki} + R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s + \frac{1}{2}X_{skis}. \tag{2.16}$$

First, taking the twice covariant derivative in (2.8) we obtain

$$\frac{1}{2}S_{ki} = -\frac{1}{2}X_{ssik} + \frac{n}{2}\lambda_{ik},$$

comparing with (2.16), (2.2), (2.3) and (2.5) we have

$$\begin{aligned}
Y &= -\frac{1}{2}X_{sski} + \frac{n}{2}\lambda_{ik} + R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s + \frac{1}{2}X_{skis} \\
&= \frac{1}{2}(X_{skis} - X_{sski}) + R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s + \frac{n}{2}\lambda_{ik} \\
&= \frac{1}{2}(X_{skis} - X_{sksi} + X_{sksi} - X_{sski}) + R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s + \frac{n}{2}\lambda_{ik} \\
&= \frac{1}{2}(R_{tsis}X_{tk} + R_{tkis}X_{st} + R_{tsks}X_{ti} + R_{tsks,i}X_t) \\
&+ R_{sk}R_{si} - \frac{1}{2}R_{is,k}X_s + \frac{n}{2}\lambda_{ik} \\
&= \frac{1}{2}(R_{ti}X_{tk} + R_{tkis}X_{st} + R_{tk}X_{ti}) + \frac{1}{2}(R_{sk,i} - R_{si,k})X_s \\
&+ R_{sk}R_{si} + \frac{n}{2}\lambda_{ik} \\
&= \frac{1}{2}(R_{si}X_{sk} + R_{tkis}X_{st} + R_{sk}X_{si}) - \frac{1}{2}R_{tski,t}X_s + R_{sk}R_{si} + \frac{n}{2}\lambda_{ik} \\
&= R_{si}(R_{sk} + \frac{1}{2}X_{sk}) + \frac{1}{2}(R_{tkis}X_{st} + R_{sk}X_{si}) - \frac{1}{2}R_{tski,t}X_s + \frac{n}{2}\lambda_{ik} \\
&= R_{si}(-\frac{1}{2}X_{ks} + \lambda g_{sk}) + \frac{1}{2}(R_{tkis}X_{st} + R_{sk}X_{si}) - \frac{1}{2}R_{tski,t}X_s + \frac{n}{2}\lambda_{ik} \\
&= -\frac{1}{2}R_{si}X_{ks} + \lambda R_{ik} + \frac{1}{2}R_{tkis}X_{st} + \frac{1}{2}R_{sk}X_{si} - \frac{1}{2}R_{tski,t}X_s + \frac{n}{2}\lambda_{ik}.
\end{aligned}$$

Substituting the previous expression in (2.15) we deduce

$$\begin{aligned}
\Delta R_{ik} &= -\frac{1}{2}R_{si}X_{ks} + 2\lambda R_{ik} + \frac{1}{2}R_{tkis}X_{st} + \frac{1}{2}R_{sk}X_{si} \\
&- \frac{1}{2}R_{tski,t}X_s + 2R_{skij}R_{sj} + \frac{1}{2}R_{skij}X_{js} + \frac{1}{2}R_{ik,s}X_s \\
&- \frac{1}{2}X_{ikss} + \lambda_{jj}g_{ki} - \lambda_{ij}g_{kj} + \frac{n}{2}\lambda_{ik}.
\end{aligned}$$

Now from (2.2) and (2.3) we have

$$\begin{aligned} X_{ikss} - X_{issk} &= X_{ikss} - X_{isks} + X_{isks} - X_{issk} \\ &= R_{tiks}X_{ts} + R_{tiks,s}X_t + R_{tiks}X_{ts} + R_{tsks}X_{it}. \end{aligned}$$

On the other hand, taking the covariant derivative in (2.9) we have

$$X_{issk} = -R_{ti,k}X_t - R_{ti}X_{tk} - (n-2)\lambda_{ik}.$$

Thus,

$$X_{ikss} = -R_{ti,k}X_t - R_{ti}X_{tk} + R_{tiks}X_{ts} + R_{tiks,s}X_t + R_{tiks}X_{ts} + R_{tk}X_{it} - (n-2)\lambda_{ik}.$$

Finally, we may use the first Bianchi identity and (2.5) to infer

$$\begin{aligned} \Delta R_{ik} &= 2\lambda R_{ik} - 2R_{ijks}R_{js} - \frac{1}{2}R_{si}X_{ks} + \frac{1}{2}R_{tkis}X_{st} \\ &+ \frac{1}{2}R_{sk}X_{si} - \frac{1}{2}R_{tski,t}X_s + \frac{1}{2}R_{skij}X_{js} + \frac{1}{2}R_{ik,s}X_s \\ &- \frac{1}{2}(-R_{it,k}X_t - R_{it}X_{tk} + R_{tiks}X_{ts} + R_{tiks,s}X_t + R_{tiks}X_{ts} + R_{tk}X_{it}) \\ &+ \frac{(n-2)}{2}\lambda_{ik} + \lambda_{jj}g_{ki} - \lambda_{ij}g_{kj} + \frac{n}{2}\lambda_{ik} \\ &= 2\lambda R_{ik} - 2R_{ijks}R_{js} - \frac{1}{2}R_{si}X_{ks} + \frac{1}{2}R_{tkis}X_{st} + \frac{1}{2}R_{sk}X_{si} \\ &- \frac{1}{2}R_{tski,t}X_s + \frac{1}{2}R_{skij}X_{js} + \frac{1}{2}R_{ik,s}X_s + \frac{1}{2}R_{is,k}X_s + \frac{1}{2}R_{is}X_{sk} \\ &- \frac{1}{2}R_{tiks}X_{ts} - \frac{1}{2}R_{tiks,s}X_t - \frac{1}{2}R_{tiks}X_{ts} - \frac{1}{2}R_{sk}X_{is} \\ &+ (n-1)\lambda_{ik} + \lambda_{jj}g_{ki} - \lambda_{ij}g_{kj} \\ &= 2\lambda R_{ik} - 2R_{ijks}R_{js} + \frac{1}{2}R_{is}(X_{sk} - X_{ks}) + \frac{1}{2}R_{sk}(X_{si} - X_{is}) \\ &+ \frac{1}{2}R_{ik,s}X_s + \frac{1}{2}R_{ik,s}X_s - \frac{1}{2}R_{tisk,t}X_s - \frac{1}{2}R_{tski,t}X_s \\ &+ R_{tkis}X_{st} - R_{tiks}X_{ts} - \frac{1}{2}R_{tiks,s}X_t + (n-1)\lambda_{ik} + \lambda_{jj}g_{ki} - \lambda_{ij}g_{kj} \\ &= 2\lambda R_{ik} - 2R_{ijks}R_{js} + R_{ik,s}X_s + \frac{1}{2}R_{is}(X_{sk} - X_{ks}) + \frac{1}{2}R_{sk}(X_{si} - X_{is}) \\ &+ R_{skit}X_{ts} - R_{tiks}X_{ts} - \frac{1}{2}R_{tisk,t}X_s - \frac{1}{2}R_{tski,t}X_s \\ &- \frac{1}{2}R_{tkis,t}X_s + (n-1)\lambda_{ik} + \lambda_{jj}g_{ki} - \lambda_{ij}g_{kj} \\ &= 2\lambda R_{ik} - 2R_{ijks}R_{js} + R_{ik,s}X_s + \frac{1}{2}R_{is}(X_{sk} - X_{ks}) + \frac{1}{2}R_{sk}(X_{si} - X_{is}) \\ &+ (n-1)\lambda_{ik} + \lambda_{jj}g_{ki} - \lambda_{ij}g_{kj}, \end{aligned}$$

which finishes the proof of the lemma. □

3. Proof of the results

3.1. **Proof of Theorem 1.**

Proof. First of all we compute the trace of identity (2.11) to obtain

$$\frac{1}{2}\Delta S - \frac{1}{2}\langle \nabla S, X\rangle = \lambda S - |Ric|^2 + (n-1)\Delta\lambda. \tag{3.1}$$

Now, using that $|Ric - \frac{S}{n}g|^2 = |Ric|^2 - \frac{S^2}{n}$, we infer

$$\frac{1}{2}(\Delta S - \langle \nabla S, X\rangle) = -|Ric - \frac{S}{n}g|^2 + \frac{S}{n}(n\lambda - S) + (n-1)\Delta\lambda. \tag{3.2}$$

On integrating identity (3.2) and using the compactness of M^n we arrive at

$$\int_M |Ric - \frac{S}{n}g|^2 dV_g = \frac{1}{2}\int_M \langle \nabla S, X\rangle dV_g + \frac{1}{n}\int_M S div X dV_g. \tag{3.3}$$

Now we recall that for any vector field Z on M^n we have

$$div(SZ) = S div Z + \langle \nabla S, Z\rangle. \tag{3.4}$$

Whence, we deduce

$$\int_M |Ric - \frac{S}{n}g|^2 dV_g = \frac{n-2}{2n}\int_M \langle \nabla S, X\rangle dV_g = -\frac{n-2}{2n}\int_M S div X dV_g, \tag{3.5}$$

which finishes the first statement. Proceeding, we notice that using once more identity (1.3) we have from (3.5) $\int_M |Ric - \frac{S}{n}g|^2 dV_g = -\frac{n-2}{2n}\int_M S\Delta h dV_g$, which completes the proof of the theorem. □

3.2. **Proof of Corollary 1.**

Proof. We point out that any assumption of the corollary jointly with Theorem 1 give $Ric = \frac{S}{n}g$. Whence we obtain $\frac{1}{2}\mathcal{L}_X g = (\lambda - \frac{S}{n})g$, i.e., X is a nontrivial conformal vector field. Now we may apply Theorem 2 in [2] to conclude that M^n is isometric to a Euclidean sphere, which finishes the proof of the corollary. □

3.3. **Proof of Corollary 2.**

Proof. First we notice that from Corollary 1 we may assume that M^n is isometric to a Euclidean sphere $\mathbb{S}^n$. In particular, using (1.1) we obtain

$$\Delta h = n\lambda - n(n-1). \tag{3.6}$$

Now we invoke identities (3.2) and (3.6) to obtain

$$\Delta(h + \lambda) = 0, \tag{3.7}$$

where $X = \nabla h + Y$ according to (1.3). Whence we have $h = -\lambda + c$, where c is a constant. Using once more (3.6) we deduce

$$\Delta h + nh = n(c - (n-1)). \tag{3.8}$$

Therefore, up to constant, h is a first eigenfunction of the Laplacian of $\mathbb{S}^n$. Consequently, ∇h is a conformal vector field. Since X is also conformal in this case, we may invoke Hodge-de Rham decomposition to deduce that Y is a Killing vector field, which completes the proof of the corollary. □

3.4. **Proof of Corollary 3.**

Proof. When X is a conformal vector field it is well known that $\int_M \langle \nabla S, X\rangle dV_g = 0$, see e.g. [3]. Hence, we deduce from Theorem 1 that $Ric = \frac{S}{n}g$. Thus we have that its scalar curvature is constant and the result follows from the last part of the proof of Corollary 2. The second assertion follows from the last argument and we complete the proof of the corollary.

□

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